Special Workshop on Risk Acceptance and Risk Communication
March 26-27, 2007, Stanford University

Cost and benefit including value of life and limb measured in time units

Ove Ditlevsen and **Peter Friis-Hansen**
Technical University of Denmark
Nils Koppels Allé, Building 403, DK 2800 Kgs. Lyngby, Denmark, od@mek.dtu.dk, pfh@mek.dtu.dk

Abstract: Key elements of the authors' work on money equivalent time allocation to costs and benefits in risk analysis are put together as an entity. This includes the data supported dimensionless analysis of an equilibrium relation between total population work time and gross domestic product leading to the definition of the life quality time allocation index (LQTAI). A postulate of invariance of the LQTAI allocates societal value in terms of time to avoid life shortening fatalities as well as serious injuries that shorten the life in good health. The observed large difference between the actual costs used by the owner for economically optimizing an activity motivates a simple risk accept criterion suited to be imposed on the owner by the public. The paper ends with an example concerning allocation of economical means for mitigation of loss of life and limb on a ferry in fire.

1. Introduction

An engineering activity is planned to be realized within a statistically homogeneous economical region. By dividing all costs and benefits in the expected net cost equation by the average wage per time unit over the population of the region the equation gets physical dimension as time, and the equation becomes independent of local inflation an purchase power.

The equation may contain terms that represent loss of life and limb, and possibly also environmental damages. Such terms have in the past been considered as intangibles causing them to be excluded from the cost-benefit analysis. However, ideas based on macro economical reasoning have in the last decade opened possibilities of making rational evaluations (e.g. by use of the life quality index (LQI) defined in (Nathwani, Lind, & Pandey 1997, Pandey, Nathwani, & Lind 2006) and extensively studied and applied in (Rackwitz 2002, Rackwitz, Lentz, & Faber 2005), or, for environmental damage, the Nature preservation willingness index defined in (Friis-Hansen & Ditlevsen 2003)).

In the time formulation the life and limb losses may at a first glance simply be written as the increment of the expected life time in good health caused by the loss giving accident. However, this would be an oversimplification because a part of the loss is work time of larger societal value than the free time. The correction can be made by use of the criterion of invariance of the LQI, or, directly in time units, by use of invariance of the life quality time allocation index (LQTAI) defined in (Ditlevsen & Friis-Hansen 2005, Ditlevsen & Friis-Hansen 2007, Ditlevsen 2007). The results may be slightly different because the LQTAI is an extended more general version of the LQI. This paper gives a short recapitulation of the authors' thoughts behind the LQTAI including its empirical support. To the authors' surprise these thoughts and the supporting empirical findings have turned out to be controversial.

For large projects the decision making by the owner is restricted by public requirements. The owner is primarily focused on maximizing the profit only including the direct costs to be spent on insurance premiums, uncovered losses and damage compensations. These costs are often much less than the societal value of life and limb as obtained from the invariance of the LQI or the LQTAI. Therefore the society must consider the possibility of this larger loss and protect itself under the consideration that the society has a positive interest in the realization of the project. Rational reasoning leads to a public accept criterion formulated in (Friis-Hansen & Ditlevsen 2003, Ditlevsen 2003) and is recapitulated herein.

Finally the paper gives an example of using the LQTAI to assess the expected societal time value loss of life and limb due to a fire on a ferry.

2. Standard expected loss function example

The standard example of technical decision problem is related to an operation (ship transport, factory, structure, etc.) that in the mean produces a net gain g per time unit facing the fact that n different independent categories of cost generating accidents occur in time at random time points that for the ith category is in accordance with a homogeneous Poisson process of intensity $\lambda_i, i = 1, \dots, n$. Denoting the mean of the random cost of an accident in category i by μ_i, the expected loss function related to the operation becomes asymptotically as $t \to \infty$, (Ditlevsen 2003)

$$L(\lambda_1, \dots, \lambda_n, \mu_1, \dots, \mu_n) = c(\lambda_1, \dots, \lambda_n, \mu_1, \dots, \mu_n) + \frac{1}{\gamma}\sum_{i=1}^{n} \lambda_i \mu_i - \frac{g}{\gamma} \tag{1}$$

where $c(\lambda_1, \dots, \lambda_n, \mu_1, \dots, \mu_n)$ is the initial cost invested at time zero and γ is the interest rate. Of dimensional reasons it follows that the initial cost function must be linear in $\mu_1, \dots, \mu_n$. The unit of (1) may be the currency of the relevant country, and as such the amounts are time dependent on inflation and current purchase power of the currency in the actual country. Thus the μ-values must be corrected to equivalent values at time zero. This correction is made automatic if the current values are divided by the average current wage (salary) per time unit over the entire population of the country. Thus the mean losses μ_i and the gain g become expressed as the time it takes to earn the amounts by the average wage. To the first approximation these amounts measured in time units become independent of time and to some degree of the same size from country to country.

The owner of the operation then acts optimally if the operation is designed so that $\lambda_1, \dots, \lambda_n, \mu_1, \dots, \mu_n$ get values that minimize the loss function (1). Even though some of the μ-values may represent values of life and limb these values are not intangibles from the owners point of view. These values are simply those that are implied by common insurance premiums and damage compensation practice. These *"owner's values"* are usually considerably less than the *"societal values"* obtained from life quality index considerations (LQI) as first advocated by Nathwani, Lind, and Pandey a decade ago (Nathwani, Lind, & Pandey 1997), and since then further developed by the same authors (Pandey, Nathwani, & Lind 2006) and by Rackwitz (Rackwitz 2002, Rackwitz, Lentz, & Faber 2005).

The observation that the societal loss value is much larger than the owners loss value related to some accident causes the society to set up certain restrictions on the owner's activity to ensure that the societal loss is fully compensated by the benefit the society obtains from the owner's activity. A rational public acceptance rule based on the concept of owner's value versus societal value is formulated in (Ditlevsen 2003, Friis-Hansen & Ditlevsen 2003), and is recapitulated in Section 6.

Having a wish to better understand the first principles behind the LQI formulation, and to see to which degree the LQI may be empirically justifiable and to which degree it is based on a postulate of consistent behavior, the authors came to a slightly different and more general societal life quality model based on dimensionless formulation of preference equilibrium between work time and free time (leisure time). The formulation of this model is in short form recapitulated in the next section based on the authors' papers (Ditlevsen & Friis-Hansen 2005, Ditlevsen & Friis-Hansen 2007, Ditlevsen 2007).

3. Preference equilibrium between work time and free time in good health

This section closely follows the text in (Ditlevsen 2007) with a minor but important generalization present in the earlier paper (Ditlevsen & Friis-Hansen 2005) but not explicitly addressed in the cited papers on the LQI. The model formulation in Section 2 of (Ditlevsen & Friis-Hansen 2005) starts with random variable modeling of the relevant quantities for the individual member of the population whereupon expectations are taken over the population. Herein as in (Ditlevsen & Friis-Hansen 2007, Ditlevsen 2007) the modeling starts directly on the level of mean values over the entire population.

The slight generalization is that only a fraction r of the time unit represents time in good health while the fraction $1 - r$ represents time with illness or bad health, that is, time that cannot be used as (money-producing) work time. Thus $w \leq r$ where w is the fraction of the time unit used for work, and the ratio $w/r \leq 1$ is the fraction of the time in good health used for work.

3.1. Time value modeling

An increment $\mathrm{d}w$ has a monetary value directly proportional to the monetary value of the production in the time increment, of course. However, what herein is called "experienced worth" is reasonably defined as the dimensionless ratio of this increment value to the value of the production in the entire work time fraction w. This definition of experienced worth of work time is simply based on the psychological feeling that a small increment of "much" has less worth than a small increment of "little". The definition is consistent with the empirical Weber-Fechner law that states that there is a *logarithmic* relation between the physical magnitudes of stimuli and the perceived intensity of the stimuli (source http://en.wikipedia.org/wiki/Weber-Fechner_law). Thus the experienced worth of the increment $\mathrm{d}w$ is measured relatively as $\mathrm{d}(pw)/pw$, where p is a factor defined as the ratio between the total yearly GDP (Gross Domestic Product) G within a geographical region such as a country and the total yearly work based salary S paid out to the inhabitants of the region.

The factor p may be interpreted as an enhancement factor that increases (or decreases) the societal value of the work time fraction from w to pw. The corresponding money value is then $G = pS = pw(S/w)$, where S/w is the salary per work time unit. It is seen that p appears as an overall (or "equivalent") productivity that enhances S to G. This enhancement includes all possible sources to G, that is, also those sources not directly related to the used manpower. Basically G is a measure of the economical wealth of the society and S is a measure of how much of this economical wealth is distributed directly to the working population. In the following p will be denoted as the "time equivalent productivity" realizing that the word "productivity" is used differently in production economics. More details are discussed in (Ditlevsen 2007).

Similarly it is assumed that the experienced worth of the free time *in good health* is measured relatively as $\mathrm{d}(r-w)/(r-w)$ which for a constant r is the same as $\mathrm{d}(1-w/r)/(1-w/r)$. Note that $\mathrm{d}(1-w/r)/(1-w/r)$ is negative for a positive work fraction increment $\mathrm{d}w$. Therefore a balance between $\mathrm{d}(pw)/pw = \mathrm{d}(pw/r)/(pw/r)$ and $\mathrm{d}(1-w/r)/(1-w/r)$ may exist for which constant worth is maintained. This consideration leads to the balance equation

$$c\frac{\mathrm{d}\big(pw/r\big)}{pw/r} + (1-c)\frac{\mathrm{d}\big(1-w/r\big)}{1-w/r} = 0 \tag{2}$$

for some constant c between 0 and 1. More correctly, c should be written as a function $c(r)$ of r, but for simplicity of writing (r) is suppressed wherever it is not needed explicitly in the following. For explanation of (2) it is noted that any relation between differentials per definition is linear in the differentials. That the necessarily positive coefficients sum to one is just the result of a convenient normalization obtained by dividing the equation by the sum of the coefficients.

Assume now that the work market tries to maintain this balance whatever the values of w and p and assume that w locally may be a function of p (or, mathematically equivalent, that p locally may be a function of w).

Then (2) reduces to the simple differential equation

$$\left(\frac{c}{w/r} - \frac{1-c}{1-w/r}\right) \mathrm{d}(w/r) = -\frac{c}{p}\,\mathrm{d}p \tag{3}$$

between w/r and p. It is seen that $\mathrm{d}p = 0$ for $w/r = c$ and an elementary analysis of the variation of the sign of $\mathrm{d}p$ with w/r in the neighborhood of $w/r = c$ shows that p is minimal for $w/r = c$. The general solution to (3) is

$$p = p_{\min} \frac{(1-w/r)^{1-1/c}\,(w/r)^{-1}}{(1-c)^{1-1/c}\,c^{-1}} \tag{4}$$

where $p_{\min} = \min_w p$.

3.2. Data

The relation (4) is supported by data from different OECD countries. The numerical data for Denmark are given in full with detailed explanations in (Ditlevsen & Friis-Hansen 2005), and for the other OECD countries in diagrammatic form in (Ditlevsen & Friis-Hansen 2007) all based on the data in (OECD 2004).

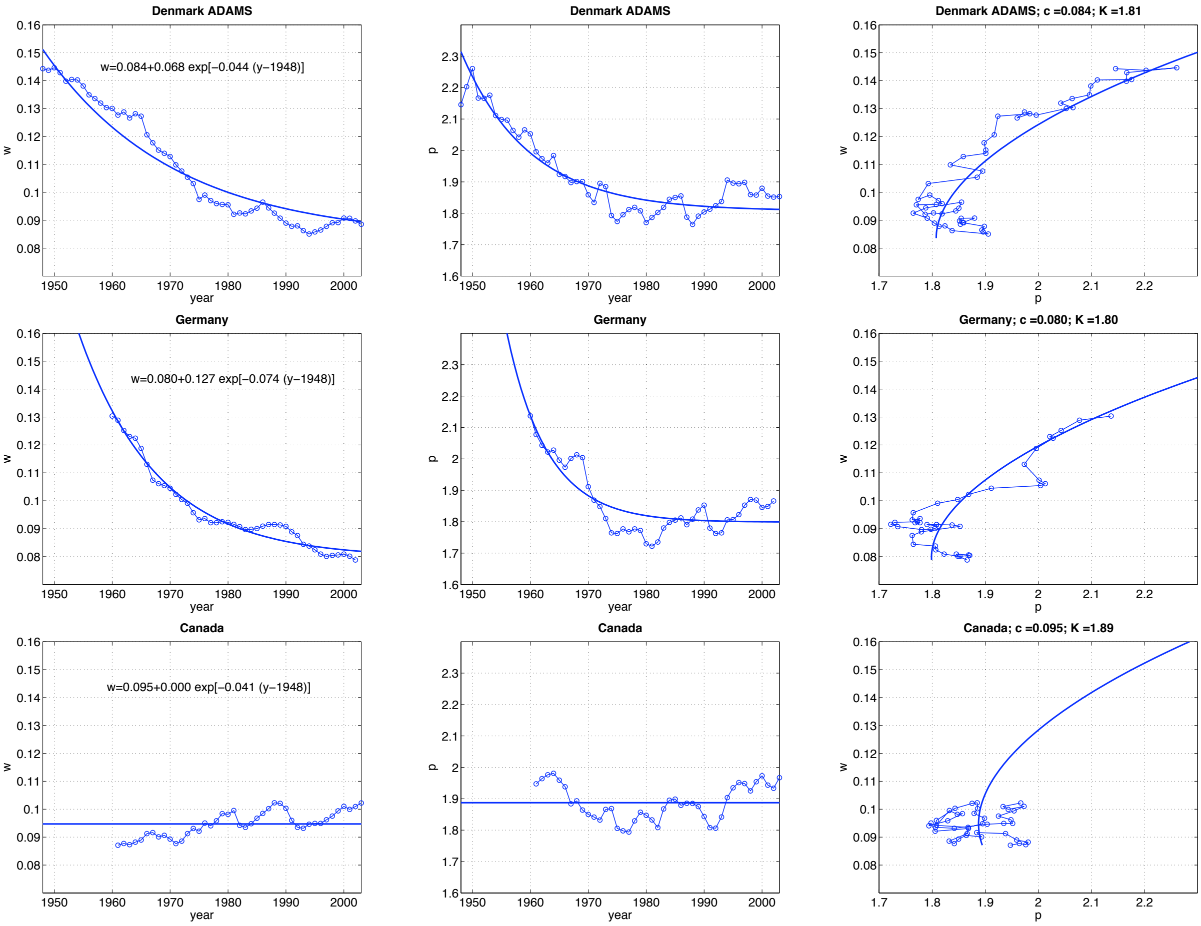

Figure 1. *Top: Data points for Denmark in the years 1948 - 2003. Recorded work time fraction* w *and time equivalent productivity* p *(ratio of salary to GDP per same time unit). The smooth curves are jointly least square fits of* (5) *(left diagram) and* (4) *(for* $r = 1$*) with* (5) *substituted (middle diagram) and are consistent with the curve in the right diagram for* (p, w) $(p_{min} = 1.81, c = 0.084)$*. Middle: Data for Germany in the years 1960 - 2003* $(p_{min} = 1.80, c = 0.080)$*. Bottom: Data for Canada in the years 1960 - 2003* $(p_{min} = 1.89, c = 0.095)$*.*

The two first top diagrams in Fig. 1 show the data for w and p as functions of the year for Denmark in the period from 1948 to 2003. The data points for the pair (p, w) are shown in the third diagram. The largest value of w corresponds to year 1948, and the connecting lines between the data points show the succession of the points with the year. The second and third row of diagrams show the corresponding data time series for Germany and Canada only reaching back to 1960, however, and therefore giving less strong support than the Danish data.

The smooth curves in the left diagrams are fits of the decreasing exponential function

$$w = c + a\, e^{-b(t-1948)} \to c \text{ as } t \to \infty \tag{5}$$

adopting the hypothesis that the asymptotic value as $t \to \infty$ is equal to the value of w at which $p(w) = p(w/1, 1)$ takes its minimal value $p_{\min}$. The unknown parameters $a, b, c, p_{\min}$ are estimated by a simultaneous least square fit of (4) and (5) to the two data sets for (t, w) and (t, p). The squared deviations are divided by the average squared deviation for each of the two data sets to bring the squared deviations into the same size scale.

In (Ditlevsen 2007) the time ordered Danish data samples of the work time ratio w and the time equivalent productivity p are analyzed down to details that make it possible to formulate a white noise driven stochastic process model for the joint time development of the two dimensionless quantities. It is remarkable that the data residuals with respect to the predicted relation as function of time show stationary random behavior in all essentials. Both processes are well described as autoregressive processes in discrete time. Simulations with the model show sample curves of great similarity to the data sample curves. Thus all significant systematic behavior of the Danish data is captured by the relation (4) for $r = 1$ and the exponential time decay (5). It is seen in all three rows of diagrams in Fig. 1 that the Danish and the German data have reached a statistically stationary state after about 1975 while a stationary state is reached before 1960 for the Canadian data. As judged by the eye, the diagrams in (Ditlevsen & Friis-Hansen 2007) for 18 OECD countries reveal that only the data for Iceland, Ireland, Japan and New Zealand seem to be deviate significantly from the predictions of the model.

3.3. The time reduction factor r and the wealth index I_r

As it is seen from (4), the time reduction factor r simply defines an affinity mapping that moves all data points (p, w) to $(p, w/r)$. The graph of (p, w) defined by (4) for $r = 1$ maps to a curve that deviates from the graph of (4). However, a simple test calculation as done in (Ditlevsen & Friis-Hansen 2007) shows that the deviation in p between the two graphs for fixed w within the relevant value domain of w is at most about 0.5% for $r > 2/3$. Since the point $(c, p_{\min})$ maps to the point $(c/r, p_{\min})$ it follows that $c(r_0) = c/r_0$ for the actual value r_0 of r. In principle the value of r_0 can be estimated through a population health investigation. A value of about 0.9 may be a reasonable assessment.

The deviations between the mapped curve and the curve defined by (4), both corresponding to r and coinciding in the point of minimum of p, are negligible as compared to the random fluctuations of the data points along the curves. Therefore it is not possible to estimate the value of r from the considered data. However, the data fits indicate that it would be in conflict with the model to claim that r has been varying substantially during the observation period.

If the differential on the left side of (2) takes a value different from zero it is a sign of unbalance between the work effort and the available free time. For a positive value of the left side the total societal worth (or societal living standard) is increasing, and for a negative value the worth is decreasing. A basic non-dimensional wealth index I_r may therefore be defined relatively by

$$\frac{\mathrm{d}I_r}{I_r} = V(r)\left[c\frac{\mathrm{d}\big(pw/r\big)}{pw/r} + (1-c)\,\frac{\mathrm{d}\big(1-w/r\big)}{1-w/r}\right] \tag{6}$$

in which $V(r)$ is some differentiable function of r. The fraction $\mathrm{d}I_r/I_r$ is taken to be a measure of the total experienced worth of work time and free time together.

Integration of (6) gives the general solution

$$I_r = J(r)\left[\left(p\frac{w}{r}\right)^c\left(1-\frac{w}{r}\right)^{1-c}\right]^{V(r)}$$

$$= J(r)\exp\left\{V(r)\left[c\log\left(p\frac{w}{r}\right)+(1-c)\log\left(1-\frac{w}{r}\right)\right]\right\} \tag{7}$$

in which $J(r)$ is another differentiable function of r. In the final definition of the wealth index I_r the two functions $V(r)$ and $J(r)$ will both be set to 1, relying on an argument of slow variation.

4. Invariance principle

Assume that some increment $\mathrm{d}(rE)$ of the expected life in good health at birth is obtained by some risk reducing action. The part $(w/r)\,\mathrm{d}(rE)$ is used for work and has the experienced worth $(pw/r)\,\mathrm{d}(rE)/(pwE) = \mathrm{d}(rE)/(rE)$, and the rest is free time that has the experienced worth $(1-w/r)\,\mathrm{d}(rE)/[(1-w/r)rE] = \mathrm{d}(rE)/(rE)$. According to the combination rule (6) the total experienced worth of the time increment $\mathrm{d}(rE)$ is then

$$\text{experienced worth of } \mathrm{d}(rE) = V(r)\frac{\mathrm{d}(rE)}{rE} = V(r)\left[\frac{\mathrm{d}r}{r}+\frac{\mathrm{d}E}{E}\right] \tag{8}$$

The *invariance principle* postulates that the amount $\mathrm{d}p$ taken from p to pay for the risk reducing activity that produces the value in (8) should be such that $\mathrm{d}I_r/I_r$ under constant work effort w is balanced by the experienced worth of $\mathrm{d}(rE)$. By differentiation of (7) under constant w it follows that

$$\frac{\mathrm{d}I_r}{I_r} = V(r)\,c\,\frac{\mathrm{d}p}{p} + V(r)\left\{c'\log\left(\frac{pw}{r-w}\right) - c\frac{1}{r-w} + \frac{w}{r(r-w)}\right.$$

$$\left. + \frac{V'(r)}{V(r)}\left[c\log\left(p\frac{w}{r}\right)+(1-c)\log\left(1-\frac{w}{r}\right)\right]+\frac{J'(r)}{J(r)}\right\}\mathrm{d}r \tag{9}$$

where "′" means derivative with respect to r. Making the reasonable assumption that the two functions $V(r)$ and $J(r)$ are sufficiently slowly varying that the terms with $V'(r)/V(r)$ and $J'(r)/J(r)$ can be neglected, it follows by adding (8) and (9) and setting to zero that the amount $-\mathrm{d}p$ taken from p given by

$$-\frac{\mathrm{d}p}{p} = \frac{1}{c}\frac{\mathrm{d}E}{E} + \left[\frac{1-c}{r-w} + c'\log\left(\frac{pw}{r-w}\right)\right]\frac{\mathrm{d}r}{c} \tag{10}$$

balances the gain by obtaining the increments $\mathrm{d}E$ of expected life and $\mathrm{d}r$ of healthy life. For the actual value r_0 of r and the stationary state where $w = c(r_0) = c(1)/r_0$ and $p = p_{\min}$, (10) becomes

$$-\frac{\mathrm{d}p}{p_{\min}} = \frac{r_0}{c(1)}\frac{\mathrm{d}E}{E} + \left[\frac{1}{c(1)} + \frac{r_0}{c(1)}c'(r_0)\log\left(\frac{p_{\min}c(1)}{r_0-c(1)}\right)\right]\mathrm{d}r \tag{11}$$

The sum of the two equations (8) and (9) leading to (10) can be obtained directly as $\mathrm{d}(I_r rE)/(I_r rE) = 0$ under constant w. The product $I_r rE$ (setting $V(r) = 1$ and $J(r) = 1$) is called the Life Quality Time Allocation Index (LQTAI) (here revised relative to (Ditlevsen & Friis-Hansen 2005) by the factor r).

To apply (11) for value assignment to $\mathrm{d}r$ an assessment of the value of the derivative $c'(r_0)$ is needed. Data support is hardly obtainable. Thus it is necessary to make some normative choice based on some appreciable assumption. Such an assumption is used in (Ditlevsen & Friis-Hansen 2007) setting $c(r) = c/r$ and obtained as seen above by postulating that p is not influenced by variations of r (implying that the GDP and the total salary changes in the same pace with the variation of r). For the purpose to obtain $c'(r_0)$ the postulate may be relaxed to be valid only for small variations of r in the vicinity of r_0. Accepting this postulate the factor to the logarithm in (11) becomes

$$\frac{r_0}{c(1)}c'(r_0) = -\frac{1}{r_0} \tag{12}$$

The two terms on the right side of (11) are the fractions of $p_{\min}$ that the population should agree that the society invests per person for removing the accident source that causes the expected life to decrease by $\mathrm{d}E$ and the expected life in good health to decrease by $E\,\mathrm{d}r$, respectively. Dividing by the mean accident rate per person gives the equivalent time values. After multiplication by the salary per time the amounts are obtained in the actual money unit. The first amount is the so-called Implied Cost of Averting a Fatality (ICAF). The second may be called the Implied Cost of Averting an Injury (ICAI).

To apply these results in a decision problem for a specific project, the increments $\mathrm{d}E$ and $\mathrm{d}r$ must be assessed in relation to the project. This assessment is made by use of engineering modeling of the consequences of the possible accidents that are taken into account in the cost-benefit analysis. Examples are given in (Ditlevsen 2003, Ditlevsen 2004).

Even though it is based on an idea of logically consistent evaluation of experienced worth, the invariance principle applied to the wealth index I_r to obtain ICAF and ICAI is a *normative principle* because it can only be supported by subjective opinion. Of logical and ethical reasons it is argued that it is the right principle to apply, but the willingness to pay and current practice may be different. The way out is to interpret the ICAF and the ICAI as *societal values* that should enter the public accept rules as formulated in Section 6.

6. Public acceptance rules

To protect the welfare of its citizens against undue damage and exploitation the public sets certain restrictions on otherwise free money making activities. The decisions related to the design and operation of the technical installations needed for these activities are in a free society left to the owners who aim for optimizing their gains per time unit. Their cost- benefit analyses typically only include the direct costs to be spent on insurance premiums and damage compensations. These costs are often much less than the societal value of life and limb as obtained in Section 4. Therefore the society must consider the possibility of this larger loss and protect itself under the consideration that the society has a positive interest in the realization of the project. A way to ensure this is to require that the company tax yield to the society is sufficiently large to cover the loss of the society in excess of the owners direct compensation after the occurrence of the damage.

The concept of society is independent of country borders implying that in the modeling it is not important whether the tax is paid in the one or the other country within a region that may embrace several countries. The company tax rate as well as other parameters of the model should be considered as local or regional assessment parameters that may vary from region to region around the world. Accordingly the model defines a procedure that in the average over all kinds of risky operations is a guide to formulate public acceptance criteria.

Referring to (1), the net gain rate after loss due to the adverse events is $\sum_{i=1}^{n}(g_i - \lambda_i\mu_{oi})$ where μ_{oi} is the owners mean cost value of the adverse event and g_i is the gain per time unit, both associated to the ith adverse event category. The gain g_i is reasonably defined so that $g_1/\mu_{o1} = g_2/\mu_{o2} = \ldots = g_n/\mu_{on}$ and $\sum_{i=1}^{n} g_i = g$. With a company tax rate of ρ, the requirement of full compensation for damage in any single category then implies that the operation is only accepted by the public if $(g_i - \lambda_i\mu_{oi})\rho \geq \lambda_i\mu_{pi}$ where μ_{pi} is the societal cost value in excess of the owner's expected cost value μ_{oi}. The acceptance rule can be written on the dimensionless form

$$\frac{\lambda_i \mu_{\text{oi}}}{g_i} \le \left(1 + \frac{1}{\rho}\frac{\mu_{\text{p}i}}{\mu_{\text{oi}}}\right)^{-1} \tag{13}$$

It is interesting to note that acceptance by requiring that the LQI (or the LQTAI) does not decrease by the planned activity is a milder requirement than (13). In fact, it is shown in (Ditlevsen 2003) that the acceptance criterion becomes (13) with $\rho = 1$.

7. A model for determining $\mathrm{d}E$ and $\mathrm{d}r$ generated by an accident

As a reasonable simplification it is assumed that the probability that any person experiences at most one serious injury due to an accident during the life time is close enough to 1 that the possibility of more than one serious injury per person can be neglected. Moreover, it is assumed that the time from birth to the injury or the death happens can be modeled as the random variable $\min\{X, Y\}$ where X is an exponentially distributed random variable of mean κ^{-1} and Y is the random time to natural death (that is, the life time in case the source of the considered fatal event is removed). This model was considered in (Ditlevsen 2003) and generalized slightly in (Ditlevsen 2004) for the time to death. Here it is used for the time to the occurrence of serious injury also. To distinguish between death and injury the life time L and the life time in good health L_{h} are modeled as

$$L = \begin{cases} \min\{X, Y\} \\ Y \end{cases} \quad \text{and} \quad L_{\text{h}} = \begin{cases} R \min\{X, Y\} & \text{both with probability } P_{\text{f}} \\ RY - T\mathbf{1}_{X<Y} & \text{both with probability } 1 - P_{\text{f}} \end{cases} \tag{14}$$

respectively, where R is the random reduction factor, $T \le RY$ is the time it takes to recover from the injury, P_{f} is the probability that the accident is fatal given that it occurs, and $\mathbf{1}_{X<Y}$ is the indicator function for the event $X < Y$. Without deeper investigations it is assumed that R is uncorrelated with both $\max\{X, Y\}$ and Y (not excluding stochastic dependence due to a possible nonlinear relation between R and Y) and that X and Y are independent. Then the expectations become, (Ditlevsen 2003),

$$\mathrm{E}[L] = P_{\text{f}}\frac{1}{\kappa}\int_0^\infty (1 - e^{-\kappa t}) f_Y(t)\,\mathrm{d}t + (1 - P_{\text{f}})\mathrm{E}[Y] \tag{15}$$

$$\mathrm{E}[L_{\text{h}}] = r\mathrm{E}[L] - (1 - P_{\text{f}})\mathrm{E}[T]E[\mathbf{1}_{X<Y}] = (r + \mathrm{d}r)E \tag{16}$$

where $r = \mathrm{E}[R]$, $E = \mathrm{E}[L]$ and where T is assumed to be independent of X and Y. The last equality is according to the definition of the mean life in good health. It follows from (15) that

$$\frac{\mathrm{E}[Y] - \mathrm{E}[L]}{\kappa \mathrm{E}[L]^2} \to P_{\text{f}}\frac{1}{2}(1 + V_Y^2) \quad \text{or} \quad \frac{\mathrm{d}E}{E} \approx -P_{\text{f}}\frac{1}{2}(1 + V_Y^2)\,\kappa E \tag{17}$$

as $\kappa \to 0$. V_Y (≈ 0.2) is the coefficient of variation of Y. This is a simple modification of the result in (Ditlevsen 2003) corresponding to $P_{\text{f}} = 1$ and proved in the same way. Moreover it follows from (16) that

$$\mathrm{d}r = -(1 - P_{\text{f}})\kappa \mathrm{E}[T] \tag{18}$$

using that $E[\mathbf{1}_{X<Y}] = P(X < Y) = \kappa E$, (Ditlevsen 2003).

8. Distribution on accident categories

The results (17) for $\mathrm{d}E$ and (18) for $\mathrm{d}r$ are for a single accident category. Assume that it is possible to categorize all essential accidents into n different categories and that the population of size N can be assigned to these categories so that N_i persons are assigned to category $i = 1, \ldots, n$. By assumption

$N_1 + \ldots + N_n = N$ since no person can belong to two or more categories. The total p-allocation per time unit to accidents is then $N\,\mathrm{d}p = N_1(\mathrm{d}p)_1 + \ldots + N_n(\mathrm{d}p)_n$ where

$$\frac{(\mathrm{d}p)_i}{\kappa_i} = \frac{r_0}{c(1)}\frac{1}{2}(1+V_Y^2)\,Ep_{\min}P_{\mathrm{f}i} + \left[\frac{1}{c(1)} + \frac{1}{r_0}\log\left(\frac{r_0 - c(1)}{p_{\min}c(1)}\right)\right]\mathrm{E}[T_i]p_{\min}(1 - P_{\mathrm{f}i}) \quad (19)$$

according to (10), (17) and (18). Since κ_i is the mean fraction per time unit of the population associated to category i that suffers from an accident, each person associated to the ith category gets allocated a time loss equal to the right side of (19).

9. Example: Ferry on fire

This example illustrates the socio-economic assessment of life and limb by application to a risk analysis of a fire on a RoRo ferry carrying 150 passengers inclusive crew. Although fires represent a serious hazard for ships, many studies have shown that fire is the second largest hazards for crew and passengers on ships. Foundering due to collision, grounding or hull structural failure is generally considered being more dominant hazards. Nonetheless, for illustrative purposes the focus is here on assessing the number of injuries, the related duration of injury, and the number of fatalities.

The most critical phase, as regards fatalities and injuries of crew and passengers, is not measured directly after the occurrence of the incident. In the phase that follows, the captain may try to prevent the incident from evolving into a serious accident by, for example, intentionally beaching a ship that takes in water and thereby try to keep it from sinking. However, if such risk mitigating measures fail, evacuation provides the last opportunity to minimize injuries and fatalities. When an evacuation is initiated the evacuation performance is very important. Clearly an orderly and timely evacuation is crucial.

To model the critical situation, a Bayesian Network construct of the fire scenario is very useful. The network constructed for this example considers initiation of fires in the accommodation areas and in the engine room. Also the escalation of the fires is included as a model element. Simulated evacuation times reported in (Vanem & Skjong 2006) are used. The number of fatalities in the fire scenario is assessed in dependence of the available evacuation time. Similarly, the probability of a person being injured during the evacuation is modeled partly as a function of the seriousness of the incident and of the available evacuation time. The seriousness of the occurring injuries in dependence of the available evacuation time is assessed on a subjective basis although supported by the findings in (Hansen & Vinter 2003).

Results: The annual frequency of fire in Scandinavian waters is estimated to $\kappa = 1.2 \cdot 10^{-2}\ \mathrm{y}^{-1}$. Given the occurrence of a fire estimates of the probability of a fatal accident and of injury is $P_{\mathrm{f}} = 7.85 \cdot 10^{-3}$ and $P_{\mathrm{i}} = 0.214$, respectively. Thus, the probability is 0.778 that the fire ocurrence does not harm an arbitrary passenger or crew member. Conditional on injury the expected recovery time is estimated to $\mathrm{E}[T \mid \text{injury}] = 0.27$ y. Hence, $E[T] = (0.778 \cdot 0 + 0.214 \cdot 0.27)/(0.778 + 0.214) = 0.059$ y $= 22$ days. Inserting these values into (19) and using $c = 0.084$, $p_{\min} = 1.81$, $r_0 = 0.95$, $E = 80$ y, and $V_Y = 0.2$, the two terms in (19) become $\mathrm{d}p_{\mathrm{f}}/\kappa = 6.68$ y and $\mathrm{d}p_{\mathrm{i}}/\kappa = 1.34$ y. Multiplying these by $S = GDP/p_{\min} = 33,340/1.81 = 18,420$ €/y their monetary equivalents are 0.123 mill € and 0.0248 mill €, respectively. Thus the societal cost of an injury is about 20% of the cost of a fatality. For $N = 150$ passengers the total expected socio-economic loss of life and limb per fire becomes $150 \cdot (0.123 + 0.025) = 22.2$ mill €.

The owner loss expectations are estimated to 100.000 € / fatality and 4.000 € / injury. The expected owner loss due to fatalities and injuries caused by a fire occurrence is thus $\mu_{\mathrm{o}} = (7.85 \cdot 10^{-3} \cdot 0.1 + 0.214 \cdot 0.004) \cdot 150 = 0.0246$ mill €. The total annual gain before tax for the RoRo-ferry is 3.075 mill € / y of which 30% is allocated to cover the fire risk. The two sides of the acceptance inequality (13) then become

$$\text{left side:} \frac{1.2 \cdot 10^{-2} \cdot 0.0246}{0.3 \cdot 3.075} = 3.20 \cdot 10^{-3}, \quad \text{right side:} \left(1 + \frac{1}{\rho}\frac{22.2 - 0.0246}{0.0246}\right)^{-1}_{\rho=0.3} = 3.32 \cdot 10^{-3} \quad (20)$$

Thus the public acceptance criterion (13) is satisfied. For $\rho = 1$ the criterion is the so-called LQI (or LQTAI) criterion which is less restrictive than if $\rho < 1$. For $\rho = 1$ the right side of (13) becomes $1.10 \cdot 10^{-2}$.

Conclusions

A recapitulation is given of the previously published dimensionless analysis of the equilibrium relation between the gross domestic product, the salary, and the work time all on average over the entire population of a macro economical homogeneous geographical region. The analysis suggests to measure all monetary quantities in the unit of time and thereby to eliminate the influence of inflation, the currency unit, and the local purchase power. The theoretically obtained work and leisure time distributing equilibrium principle has an empirical back-up that makes it reasonable also to use this equilibrium principle to define a normative rule for allocating time to life and limb saving measures in risky activities. The allocation rule is generated from invariance of the life quality time allocation index (LQTAI) derived conveniently from the dimensionless analysis behind the equilibrium principle. The inspiration to this construct comes decisively from a study of the literature on the life quality index (LQI) mainly authored by Nathwani, Lind, Pandey, and Rackwitz, and a concern to get a deeper understanding of what the basis is for the LQI. The result of the study is a more general mathematical formulation than that of the LQI (in different versions). If the macro economy including the work time fraction is long time stationary the two indices are almost identical even though their background philosophies are strongly different. One important difference is that the LQTAI is defined so that it is applicable also for a non-stationary macro economy with varying work time ratio.

A result of the more general formulation is that not only a principle is obtained for socio-econnomical allocation of time (i.e. money) to fatality mitigation, but also for allocation of reasonable and ethically defendable time for injury mitigation. The concluding example with a ferry in fire illustrates the fact that there is a large difference between the damage compensation required from the owner of a risky operation and the societal value of life and limb. This difference motivates the formulation of a public acceptance rule (previously published) which is considerably more restrictive than the LQI (or LQTAI) criterion which states that the gain per time unit must be at least as large as the societal loss value per time unit as obtained by the principle of LQI invariance.

Acknowledgment

The first author thanks Otto Mønsteds Fond for conference travel support.

References

Ditlevsen, O. (2003). Decision modeling and acceptance criteria. *Structural Safety* 25:165–191.

Ditlevsen, O. (2004). Life quality index revisited. *Structural Safety* 26:443–451.

Ditlevsen, O. (2007). Model of observed stochastic balance between work and free time supporting the LQTAI definition. *Structural Safety.* In press. Preprint on http://www.mek.dtu.dk/staff/od/papers.htm until publication.

Ditlevsen, O. & P. Friis-Hansen (2005). Life quality allocation index–an equilibrium economy consistent version of the current life quality index. *Structural Safety* 27:262–275.

Ditlevsen, O. & P. Friis-Hansen (2007). Life quality index – an empirical or a normative concept? *International Journal of Risk Assessment and Management,* in press.

Friis-Hansen, P. & O. Ditlevsen (2003). Nature preservation acceptance model applied to tanker oil spill simulations. *Structural Safety* 25:1–34.

Hansen, H. L. & M. Vinter (2003). Occupational accidents and ship design: Implications for prevention. In *World Maritime Conference, San Francisco*.

Nathwani, J. S., N. C. Lind, & M. D. Pandey (1997). *Affordable Safety By Choice: The Life Quality Method*. Waterloo, Ontario, Canada: Institute for risk Research, University of Waterloo.

OECD, S. (2004). *OECD Economic Outlook Database*, new.sourceoecd.org/database/oecdeconomicoutlook.

Pandey, M. D., J. S. Nathwani, & N. C. Lind (2006). The derivation and calibration of the life quality index (LQI) from economical principles. *Structural Safety* 28:341–360.

Rackwitz, R. (2002). Optimization and risk acceptability based on the life quality index. *Structural Safety* 24:297–332.

Rackwitz, R., A. Lentz, & M. Faber (2005). Socio-economically sustainable civil engineering infrastructures by optimization. *Structural Safety* 27:187–229.

Vanem, E. & R. Skjong (2006). Designing for safety in passenger ships utilizing advanced evacuation analysis - a risk based approach. *Safety Science* 44:111–135.