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Risk Communication with Generalized Uncertainty and Linguistics

Ross B. Corotis
University of Colorado
428 UCB, Boulder, CO 80309-0428

Abstract

Civil Engineers have the opportunity and obligation to lead society to more effective decision-making for built environment risk trade-offs. This paper addresses the gap between classical mathematical analysis and the linguistic-based issues and factors that play a major role on societal decisions.

A large stumbling block is the utilization of the fairly extensive literature in social psychology related to risk avoidance, in formal mathematical decision frameworks based on probabilistic analysis. Fundamental principles of generalized information theory may be helpful in casting sociological considerations of perceived risk into linguistic frameworks so that the mathematics of information theory can be applied to develop decision guidelines. Fuzzy set theory is one example where probability-based uncertainty has been broadened to incorporate linguistic input. Other examples are monotone measures, such as Möbius representations, imprecise probabilities and decision weights, as well as Shannon entropy.

1. Introduction

There is a need for a definitive evaluation of the level of acceptable risk to society for the built infrastructure. Such an evaluation is necessary in order to assess the optimal trade-off of hazard prevention benefit and costs. It is necessary in such an evaluation to call upon the extensive literature in the field of social psychology related to risk avoidance. It is also necessary to incorporate these sociological results into a formal mathematical decision framework based on uncertainty analysis, which traditionally means probabilistic analysis. It is proposed that we utilize the fundamental principles of generalized information theory rather than just classical probability theory in order to utilize sociological considerations of perceived risk, which has generally been expressed in linguistic frameworks. With this approach the broader and more general mathematics of information theory can be applied to develop decision guidelines.

2. Generalized Uncertainty

2.1 Philosophy

Classical probability theory is very powerful and can address many situations in the decision rhetoric. When combined with utility theory, it forms the basis of optimal decision theory, developed over 50 years ago (von Neumann and Morgenstern, 1944). For practical societal decisions, it has been convenient to distinguish various sources of uncertainty, recognizing that a deeper understanding can be obtained by not combining all sources of uncertainty into one probability measure. For this reason, hazards analysis should distinguish between epistemic and aleatory uncertainty, reflecting such distinctions as inherent randomness, parameter uncertainty, and even model approximations and shortcomings (Corotis, 2003a; Ellingwood, 2001). For many sociological applications of uncertainty it has been necessary to extend classical probability theory to fuzzy logic, incorporating linguistic variability. For other cases, the use of subjective probability, which pre-dates frequentist definitions but was out of favor for many decades, is necessary to capture uncertainty (Vick, 2002). Yet these modifications of classical probability theory still often fail to represent the total degree of uncertainty in many practical situations (hence, for instance, the title of a session at the American Concrete Institute 2006 Fall Convention: *How Reliable is Reliability?*).

2.2 Uncertainty Measurement

When converting linguistic information into uncertainty measures, it is necessary to introduce generalized measures, or what Klir (2006) calls imprecise probabilities. The theory of such measures is based on generalizing the classical countable additivity characteristic of unions of sets with a weaker measure, maintaining only monotonicity. These so-called monotone measures (Klir, 2006), permit the inclusion of concepts such as fuzzy set membership functions and measurement error end point discount functions. This enables a generalization of such standard probabilistic axioms as the union relationship of disjoint sets,

$$P[A \cup b] = P[A] + P[B] \quad (1)$$

with a monotone measure, μ, such that

$$\mu[A \cup B] \leq\equiv\geq \mu[A] + \mu[B] \quad (2)$$

depending on the interactivity of A and B. The term μ is a monotone measure, that in classical probability would be defined by μ(Ø) = 0 and μ(X) = 1, in which Ø is a null or empty set. In its more general form, the first condition still must hold, but the second is replaced by the more general relationship

$$\forall \quad A, B \in C, \quad if \quad A \subseteq B, \quad then \quad \mu[A] \leq \mu[B] \quad (3)$$

This simple generalization allows the replacement of precise real numbers of probability with imprecise numbers, reflecting such issues as multiply possible probability distributions and uncertain calibration parameters. This introduction of imprecise probabilities allows the use, for instance, of multiple joint distributions when only the marginal distributions of the variables are known from observation. A few illustrations are given later in the proposal.

2.3 Fuzzy Sets

While the full theorems of generalized information theory are quite complex, and many have not been substantiated with data, the basic concepts should be sufficient in this research to bring in the concepts of linguistic variables. For instance, standard fuzzy set theory utilizes membership functions that are characteristics of variables (Zadeh, 1965), and it is possible to apply these same principles to generalized monotone measures of variables. Fuzzy sets are very important in the conversion of linguistic variables, such as large, moderate, and low perceived risk. As will be seen later, the concepts of risk perception have been used by social psychologists to evaluate “true” sociological risk, defined by normative theory. This information has been elucidated by sociological professionals, and needs to be converted into mathematical meaning.

The guiding equations of fuzzy sets and fuzzy logic have been developed over the past few decades. A convenient and commonly utilized basic concept of fuzzy sets is founded on a membership level, α, such that a fuzzy set A is defined to be contained within a set X as long as the membership levels for all elements within A are at least as great as α. In equation form this gives

$$A = \langle x | A(x) \geq \alpha \rangle \tag{4}$$

This is referred to as the cut-set definition of set membership, and it can be used to extend classical Boolean-based theory to fuzzy set theory. It replaces the algebra of fuzzy sets with the algebra of classical probability theory since the cut-set approach defines a subset of crisp sets.

Other definitions of fuzzy algebra are possible, as long as fuzzy sets reduce to crisp sets when the membership functions are reduced to simple (0,1) values. It is in this realm of more general operations that we can expect to find new applications that are appropriate for the conversion of linguistic variables into the mathematical realm of general uncertainty theory. It will be necessary to distinguish between uncertainties in the linguistic definitions (what is a large perceived risk?) from those associated with a lack of data or of measurement.

The tenets of probability theory can be applied to fuzzy logic by extending the mathematical rigor from crisp sets to fuzzy sets. Because of the non-exclusive nature of membership functions, however, many of the relations of classical probability theory are only preserved as inequalities. It is due to this limitation that additional measures of uncertainty need to be employed to utilize the full power of fuzzy sets.

The basic calculus of fuzzy sets can easily be seen by starting first with the following basic relationship for crisp sets

$$E[Y] = \int g(x) f_X(x) dx \tag{5}$$

in which $Y = g(X)$, and the integral is assumed to be over all possible values of X. This can be generalized to the probability of existence of a particular set, A, by replacing $g(x)$ by the indicator function, $I(x)$. Then

$$P[Y] = \int I(x) f_X(x) dx \tag{6}$$

and this relationship can be adapted for a fuzzy set, A, by replacing the indicator function by the membership function, $A(x)$, yielding

$$P[A] = \int A(x) f_X(x) dx \tag{7}$$

This simple substitution of fuzzy sets for crisp sets preserves basic classical probability measures, such as

$$P[A \cup B] = P[A] + P[B] - P[A \cap B] \tag{8}$$

in which A and B are fuzzy sets. But other properties are not preserved for fuzzy sets, so

$$P[A \cup \bar{A}] \leq 1 \quad \text{and} \quad P[A \cap \bar{A}] \geq 0 \tag{9}$$

Because of this, it is necessary to generalize probability theory into uncertainty theory first, and then to apply fuzzy logic to the generalized uncertainty measures. This approach starts with replacing probability be some monotone measure, μ.

2.4 Alternative Measures of Uncertainty

Monotonic, rather than linear, measures of uncertainty give rise to such concepts as gradations of belief in the membership measure, and these can easily be applied to fuzzy sets. For instance, one can define a belief measure, such that the belief associated with a union of sets has a lower bound of the sums of beliefs of the elements of the set. Klir (2006) defines this is a weaker version of the additive property of classical probability measures. Thus, traditional probability density functions are characteristics defined for specific values of a variable, X, while probability functions in the generalized theory are defined on the degree of belief. While the integral of a probability density function over all values must equal unity, the integral of belief measures can be greater than, equal to, or less than one. When the integral is greater than one, the belief measure is termed an upper probability, whereas when it is less than one, the belief measure is termed a lower probability. The use of generalized belief measures is not the same as fuzzy logic. In fact, belief measures can be applied to crisp sets, and they can also be applied to fuzzy sets, in which case upper and lower probabilities (or a probability interval) can be defined to a fuzzy set or a fuzzy interval.

A very powerful generalization of classical probability theory is based on Shannon's entropy (Shannon, 1948). Over fifty years ago, Shannon showed that the only logical form for entropy in a probabilistic application was given by

$$H = -a\int f_X(x)\log_b(x)dx \tag{10}$$

where a and b are positive constants selected to allow the entropy, H, to be calibrated as desired. For mathematical convenience, we will let $a = 1$ and $log_b = ln$, even though this removes the flexibility of calibration. The use of entropy then allows generalized uncertainty or information guidelines to be based on the principles of minimum and maximum entropy, which can be viewed as measures of uncertainty. It is exactly this approach that allows us the possibility of seeking solutions that invoke maximum value from a set of linguistic, or uncertain, variables. For instance, with linguistic variables obtained from multiple experts, we undoubtedly find conflict among the experts. The principle of minimum entropy defines guidelines to combine somewhat inconsistent sets with the least loss of information. On the other hand, when we are dealing with issues that may be outside of our model, such as those that might be associated with intentional disasters, the principle of maximum entropy can provide guidelines to help us bound the unknown.

3. Risk and Risk Perception Issues

3.1 Risk Factors for the Built Environment

Social psychologists have studied the perceptions of individuals to the risks they face. Some of the most complete studies have been conducted by Paul Slovic (2000), who has concluded that probability of occurrence and quantifiable consequence are only two of many factors that individuals recognize. His factor analysis provides strong evidence that many other characteristics, such as dread, voluntariness, trust and equitability drive decisions of individuals with respect to their actions. If structural engineering professionals are to be effective with their analyses, they will need to understand the bases by which the public at large and elected officials in particular, make decisions with respect to risk management (Corotis 2002, 2003b, 2004).

Some of the decisions made by the public are due to a lack of understanding of the true probabilities of occurrence (Kahneman and Tversky 2000), while others are due to aspects of the risks that are generally not incorporated by structural risk engineers, such as the role of social discounting and life quality as opposed to simple economic discount rates (Ditlevsen, 2003, 2004; Haimes, 1998, 2004; Pandey and Nathwani 2003, 2004).

3.2 Hazards and Their relationship to Activities

Slovic's research was based on 90 identified hazards to society. For the purpose of the built environment it is appropriate to identify a relevant subset. It is suggested here that 12 of the 90 activities are directly relevant to the physical environment in communities. Risks associated with transportation have been excluded from this list since they are listed separately. These 12 activities, in order of decreasing perceived risk, are: Asbestos, Liquid natural gas, Microwave ovens, Power lawn mowers, Home power tools, Dams, Home gas furnaces, Christmas tree lights, Bridges, Home appliances, Skyscrapers and Fluorescent lights. Table 1 presents the summary risk and benefit statistics for all 90 hazards, for the physical environment subset, for transportation-related hazards, and for other activities.

Table 1. Perceived Risk and Benefit for Common Hazards, on a scale of 0-100.
(adapted from Slovic, 2000)

Hazard Category	Number	Perceived Risk		Perceived Benefit	
		Average	Std Dev	Average	Std Dev
All Hazards	90	39	17	46	17
Physical Environment	12	33	9	51	12
Transportation	8	35	10	51	15
Power Generation	5	36	22	60	15
Health	35	44	15	43	16
Society	7	70	8	24	19
Job Related	4	41	8	62	23
Science	4	33	10	45	9
Recreation	15	25	5	45	10

It can be seen that the perceived risk for activities associated with the physical built environment are the lowest of all groups except for recreation (which due to its image as a highly voluntary and enjoyable activity is judged differently). It can also be seen that the perceived benefits of the physical built environment are among the highest (exceeded only by power generation and job-related activities). Therefore, the public expects the work of structural engineers to be beneficial to society and to carry a low level of risk. This is important as one seeks to determine acceptable levels of risk, since expectations are an important factor.

Power generation was not included in the physical environment category, although it could have been. Its perceived benefit is seen to be very high, and its moderate average perceived risk masks a very high spread (the highest standard deviaton of risk by far). This is due largely to nuclear power, which has a perceived risk of 72, whereas the other four sources of power have an average perceived risk of 27.

3.3 Issues Critical for Built Environment Decisions

Effective decisions with respect to risk management of natural and induced hazards on buildings and infrastructure require a consideration of all aspects of risk, including risk perception (Corotis, 2005a, b). This is especially important if decisions are to be made not just among various types of structures and infrastructure systems, but also with respect to calibration among broad risks faced by society. From Table 1 it is possible to consider the multiple aspects of risk, but some of these do not vary significantly across structure and infrastructure choices. Therefore, while important to note for built environment risk decisions, they tend to obfuscate the picture with respect to trade-offs within the structures and infrastructure field.

3.4 Linguistic Risk Assessment

Some very interesting research has recently been conducted at the University of Stuttgart attempting to cast societal risk in linguistic terms. Using Greek mythology terms as descriptors, they have concluded that the dread of the particular risk and the epistemic uncertainty factor are important aspects by which society makes its risk decisions (Klinke and Renn, 2002). This is consistent with the earlier cited work of Slovic (2000).

Klinke and Renn introduce discourse-based strategies that address key areas of risk decision-making. These are: realism versus constructivism, public concerns as criteria, uncertainties in risk assessment, risk-based versus precaution-based management, and the integration of analytic and deliberative processes. They define five components of uncertainty, which are probability, variability, systematic/random measurement errors, indeterminacy, and lack of knowledge. Their novel approach to risk representation and management consists of irregularly shaped, sometimes overlapping regions on a Cartesian set of axes of “Probability of occurrence” and “Extent of damage.” For instance, they define Damocles as a region of low probability and high consequence, such as nuclear energy, chemical facilities and dams. Cassandra, on the other hand, consists of relatively higher probability and still high consequence events, but with a long time interval prior to damages, such as anthropogenic climate change. Issues in each of these regions require different decision tree approaches, in order to satisfactorily manage complexity, uncertainty and ambiguity. They describe analytical and deliberative strategies that are based on different linguistic approaches to discourse (internal, cognitive, reflective and participatory). Their approach has been given wide consideration within the European Union and its precautionary principle (Klinke et al, 2001).

4. Generalized Uncertainty and Linguistics

The following examples and approach are taken from data in Klir (2006). They have been modified and expanded to illustrate logic that might be taken for built environment risk decisions.

Consider a West Coast U.S. city that is concerned about urban disasters, and in particular in identifying the most important source of potential hazards. Assume that the next major disaster could come from a geophysical source (e.g., earthquake), a climatological source (e.g., flood) or an intentional hazard (e.g., terrorist bomb). Assume further that the conditional true risks are 50%, 50% and 0%, respectively, but that this information is not known to the risk managers. Designating these sources as Geophysical, Climatological and Intentional, the second column in Table 6 can be created by classical probability rules, and is restricted to these values.

Table 2. Uncertainty Measures for Hazard Sources

Source	Probability	Expert Panel 1		Expert Panel 2		Combined Est.	
		B	**m**	**B**	**m**	**B**	**M**
G	0.5	5%	5%	15%	15%	21%	21%
C	0.5	5%	5%	5%	5%	9%	9%
I	0.0	0%	0%	0%	0%	1%	1%
GUC	1.0	20%	10%	40%	20%	50%	20%
GUI	0.5	20%	15%	20%	5%	34%	12%
CUI	0.5	10%	5%	10%	5%	16%	6%
GUCUI	1.0	100%	60%	100%	50%	100%	31%

Now suppose that in order to obtain estimates for the source of the next disaster, two sets of expert panels are convened. Those experts are asked to state their belief for each of the events or combinations of events shown in the table. Because these are belief measures, and not probabilities, they are generalized monotone measures and not subject to the Boolean algebra of classical probabilities. These beliefs for each panel are shown in the table in the columns marked B, and meet the rules for beliefs. The belief is a measure mapped into [0,1], with the requirement that an impossible event has B=0 and a certain event has B=1. The basic relationship of set beliefs is given by,

$$B(\cup A_i) \geq \sum_j B(A_j) - \sum_{j<k} B(A_j \cap A_k) + ... + (-1)^{n+k} B(\cap A_i) \quad (11)$$

Note that the inequality reduces to equality for classical probability measures. One of the consequences of Equation 11 is the following,

$$B(A) + B(\bar{A}) \leq 1 \quad (12)$$

Because of these inequalities, the linguistic expressions of belief are not restricted to the classical probability algebra. There is, however, a rigid set of rules, a pertinent one here is

$$B(A \cup C) = \max\{1 - B(\bar{A}), 1 - B(\bar{C})\} \quad (13)$$

With the beliefs for each set of possible outcomes it is uniquely possible to define the values in the columns labeled 'm', from the following condition:

$$B(A) = \sum_{D | D \subseteq A} m(D) \quad (14)$$

Thus, for each panel, for example, m(*G*) = B(*G*), m(*C*) = B(*C*), m(*I*) = B(*I*), and m(*G*) + m(*C*) + m(*G*U*C*) = B(*G*U*C*), etc.

The terms m(*D*) are in fact a powerful set representation known as a Möbius representation, and derived as follows. For any set of events, *A*, if $B \subseteq A$, then *B* is defined as a subset of *A*. Any monotone measure of A, $\mu(A)$ [of which $P(A)$ is the classic probability case], is uniquely related to another set, the Möbius representation, by

$$m(D) = \sum_{A|A\subseteq D} (-1)^{|D-A|} \mu(A) \tag{15}$$

In which |A-D| is the absolute value of the difference in the number of elements between sets D and A.

The Möbius representation of a set follows a strictly defined set of rules for joint, marginal and conditional relations. These can be used to derive join and marginal expressions that lead to rules of knowledge combination. The most widely accepted Möbius rule for combining evidence is given by

$$m_{1,2}(E) = \frac{\sum_{A\cap D=E} m_1(A)\cdot m_2(D)}{1-c} \tag{16}$$

In which

$$c = \sum_{A\cap D=0} m_1(A)\cdot m_2(D) \tag{17}$$

Where the summation is taken over disjoint sets of A and D. This combination rule can be used to combine the m values from the two expert panels, and leads to the m column in Table 2 under Combined Estimate. Then using Equation (15), the combined estimate beliefs can be computed for each set. These then are the consensus beliefs upon which decisions can be made. The term c may be interpreted to reflect the conflict among sources.

One final example of generalized uncertainty is based on upper and lower probability limits as estimates of imprecise probabilities. Consider two sets, $X = \{x_1,x_2\}$ and $Y = \{y_1,y_2\}$, with known marginal probabilities such that $P_X(x_1) + P_X(x_2) = 1$ and $P_Y(y_1) + P_Y(y_2) = 1$. We seek the joint distribution $P_{X,Y}(x_i,y_i)$ for $i,j \in \{1,2\}$. The four constraint equations are

$$\begin{aligned}
&P_{X,Y}(x_1,y_1) + P_{X,Y}(x_1,y_2) = P_X(x_1)\\
&P_{X,Y}(x_2,y_1) + P_{X,Y}(x_2,y_2) = P_X(x_2) = 1 - P_X(x_1)\\
&P_{X,Y}(x_1,y_1) + P_{X,Y}(x_2,y_1) = P_Y(y_1)\\
&P_{X,Y}(x_1,y_2) + P_{X,Y}(x_2,y_2) = P_Y(y_2) = 1 - P_Y(y_1)
\end{aligned} \tag{18}$$

These four equations are actually dependent, since the last is the sum of the first two minus the third. Therefore, there are three independent constraint equations associated with four joint probability values, and any one joint value can be selected arbitrarily. Assume $P_{X,Y}(x_1,y_1)$ is selected as the free value. It can not be chosen within the full range [0.1] or some of the other joint values would violate that range. The constraint is given by

$$\max\{0, P_X(x_1) + P_Y(y_1)\} \le P_{X,Y}(x_1,y_1) \le \min\{P_X(x_1), P_Y(y_1)\} \tag{19}$$

We now have a family of possible joint probability distributions that are all consistent with the marginal distributions. For any outcome, A, that consists of X,Y pairs, we can compute upper and lower bounds for $P(A)$ as

$$\bar{P}(A) = \sup_D \sum_{X\in A} P_X(x) \tag{20}$$

And

$$\underline{P}(A) = \inf_D \sum_{X \in A} P_X(x) \tag{21}$$

Where *D* indicates the set of all possible consistent joint probability distributions.

We can now develop operators for the upper and lower bound probabilities, such as $\underline{P}(A|B)$ and $\overline{P}(A|B)$. And in a generalized uncertainty sense, we can replace $\overline{P}$ and $\underline{P}$ by general monotone measures, $\overline{\mu}$ and $\overline{\mu}$ and Möbius measures $\overline{m}$ and $\underline{m}$, and gain enhanced risk assessments consistent with the level of linguistic beliefs.

Approaches such as those described in this section will be valuable additions to the currently accepted approaches to evaluating linguistic estimates of risk in terms of generalized uncertainty theory for built environment risk decisions and trade-offs.

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