Cost and benefit including value of life and limb measured in time units

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Standard expected loss function example

\[ L(\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n) = c(\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n) + \frac{1}{\gamma} \sum_{i=1}^{n} \lambda_i \mu_i - \frac{g}{\gamma} \]
Preference equilibrium between work time and free time in good health

Time value modeling

\[ G = pS = pw(S/w) \]

Weber-Fechner law

logarithmic relation

\[
c(r) \frac{d(pw/r)}{pw/r} + [1 - c(r)] \frac{d(1 - w/r)}{1 - w/r} = 0
\]
\[
\left( \frac{c(r)}{w/r} - \frac{1 - c(r)}{1 - w/r} \right) \, d(w/r) = -\frac{c}{p} \, dp
\]

\[
p(w/r, r) = p_{\text{min}} \left( \frac{(1 - w/r)^{1-1/c(r)} (w/r)^{-1}}{(1 - c(r))^{1-1/c(r)} c(r)^{-1}} \right)
\]
\[ w = c + a e^{-b(t-1948)} \rightarrow c \text{ as } t \rightarrow \infty \]
The time reduction factor $r$ and the wealth index $I_r$

\[
\frac{dI_r}{I_r} = V(r) \left[ c(r) \frac{d(p \omega / r)}{p \omega / r} + [1 - c(r)] \frac{d(1 - \omega / r)}{1 - \omega / r} \right]
\]

(6)

\[
I_r = J(r) \left[ \left( p \frac{\omega}{r} \right)^{c(r)} \left( 1 - \frac{\omega}{r} \right)^{1-c(r)} \right]^{V(r)}
\]

= $J(r) \exp \left\{ V(r) \left[ c(r) \log \left( p \frac{\omega}{r} \right) + [1 - c(r)] \log \left( 1 - \frac{\omega}{r} \right) \right] \right\}$

(7)
\[ c(r_0) = \frac{c}{r_0} \]
Invariance principle

experienced worth of \( d(rE) = V(r) \frac{d(rE)}{rE} = V(r) \left[ \frac{dr}{r} + \frac{dE}{E} \right] \) \hspace{1cm} (8)

differentiation of (7) under constant \( \omega \)

\[
\frac{dI_r}{I_r} = V(r)c(r) \frac{dp}{p} + V(r) \left\{ c'(r) \log \left( \frac{p\omega}{r - \omega} \right) - c(r) \frac{1}{r - \omega} + \frac{\omega}{r(r - \omega)} \right. \\
+ \left. \frac{V'(r)}{V(r)} \left[ c(r) \log \left( \frac{p\omega}{r} \right) + [1 - c(r)] \log \left( 1 - \frac{\omega}{r} \right) \right] + \frac{J'(r)}{J(r)} \right\} \, dr \hspace{1cm} (9)
\]

adding (8) and (9) and setting to zero

\[-\frac{dp}{p} = \frac{1}{c(r)} \frac{dE}{E} + \left[ \frac{1 - c(r)}{r - \omega} + c'(r) \log \left( \frac{p\omega}{r - \omega} \right) \right] \frac{dr}{c(r)} \hspace{1cm} (10)\]

\[
d(I_r r E)/(I_r r E) = 0
\]

Life Quality Time Allocation Index (LQTAI) \( I_r r E \)
For the actual value $r_0$ of $r$ and the stationary state where $\omega = c$ and $p = p_{\text{min}}$

$$-\frac{dp}{p_{\text{min}}} = \frac{r_0}{c} \frac{dE}{E} + \left[ \frac{1}{c} + \frac{r_0}{c} c'(r_0) \log \left( \frac{p_{\text{min}} c}{r_0 - c} \right) \right] dr$$

$$\frac{r_0}{c} c'(r_0) = -\frac{1}{r_0}$$

postulating that $p$ is not influenced by variations of $r$
Public acceptance rules

\[ \sum_{i=1}^{n} (g_i - \lambda_i \mu_{oi}) \]

\[ \sum_{i=1}^{n} g_i = g \]

\[ \frac{g_1}{\mu_{o1}} = \frac{g_2}{\mu_{o2}} = \ldots = \frac{g_n}{\mu_{on}} \]

\[ (g_i - \lambda_i \mu_{oi}) \rho \geq \lambda_i \mu_{pi} \]

\[ \frac{\lambda_i \mu_{oi}}{g_i} \leq \left( 1 + \frac{1}{\rho \mu_{oi}} \frac{\mu_{pi}}{\rho \mu_{oi}} \right)^{-1} \]
A model for determining $dE$ and $dr$ generated by an accident

$L = \begin{cases} \min\{X, Y\} & \text{and} \\ Y & \end{cases}$

$L_h = \begin{cases} R \min\{X, Y\} & \text{both with probability } P_f \\ RY - T \mathbb{1}_{X < Y} & \text{both with probability } 1 - P_f \end{cases}$

$E[\mathbb{1}_{X < Y}] = P(X < Y) = \kappa E$

$E[L] = P_f \frac{1}{\kappa} \int_0^\infty (1 - e^{-\kappa t}) f_Y(t) \, dt + (1 - P_f) E[Y]$

$E[L_h] = r E[L] - (1 - P_f) E[T] E[\mathbb{1}_{X < Y}] = (r + dr) E$

$\frac{E[Y] - E[L]}{\kappa E[L]^2} \rightarrow P_f \frac{1}{2} (1 + V_Y^2)$ or $\frac{dE}{E} \approx -P_f \frac{1}{2} (1 + V_Y^2) \kappa E$

$dr = -(1 - P_f) \kappa E[T]$
Distribution on accident categories

\( N_i \) persons are assigned to category \( i = 1, \ldots, n \)

\[ N_1 + \ldots + N_n = N \]

\[ N \, dp = N_1 (dP)_1 + \ldots + N_n (dP)_n \]

time allocation per accident suffering person in category \( i \)

\[
\frac{(dp)_i}{\kappa_i} = \frac{r_0}{c} \frac{1}{2} (1 + V_Y^2) E \rho_{\text{min}} P_{fi} + \left[ \frac{1}{c} + \frac{1}{r_0} \log \left( \frac{r_0 - c}{\rho_{\text{min}} c} \right) \right] E[T_i] \rho_{\text{min}} (1 - P_{fi})
\]
Example: Ferry on fire

\[ \frac{\lambda_i \mu_{oi}}{g_i} \leq \left( 1 + \frac{1}{\rho} \frac{\mu_{pi}}{\mu_{oi}} \right)^{-1} \]

\[ \frac{1.2 \cdot 10^{-2} \cdot 0.0246}{0.3 \cdot 3.075} = 3.20 \cdot 10^{-3} \leq \left( 1 + \frac{1}{\rho} \frac{22.2 - 0.0246}{0.0246} \right)^{-1} = 3.32 \cdot 10^{-3} \]
waste of your valuable time?
Results: The annual frequency of fire in Scandinavian waters is estimated to $\kappa = 1.2 \cdot 10^{-2} \text{y}^{-1}$. Given the occurrence of a fire estimates of the probability of a fatal accident and of injury is $P_f = 7.85 \cdot 10^{-3}$ and $P_i = 0.214$, respectively. Thus, the probability is 0.778 that the fire occurrence does not harm an arbitrary passenger or crew member. Conditional on injury the expected recovery time is estimated to $E[T \mid \text{injury}] = 0.27 \text{y}$. Hence, $E[T] = (0.778 \cdot 0 + 0.214 \cdot 0.27)/(0.778 + 0.214) = 0.059 \text{y} = 22 \text{days}$. Inserting these values into (19) and using $c = 0.084$, $p_{\text{min}} = 1.81$, $r_0 = 0.95$, $E = 80 \text{y}$, and $V_Y = 0.2$, the two terms in (19) become $d p_f / \kappa = 6.68 \text{y}$ and $d p_i / \kappa = 1.34 \text{y}$. Multiplying these by $S = GDP / p_{\text{min}} = 33,340 / 1.81 = 18,420 \text{€/y}$ their monetary equivalents are 0.123 mill € and 0.0248 mill €, respectively. Thus the societal cost of an injury is about 20% of the cost of a fatality. For $N = 150$ passengers the total expected socio-economic loss of life and limb per fire becomes $150 \cdot (0.123 + 0.025) = 22.2 \text{mill €}$.

The owner loss expectations are estimated to $100,000 \text{€/f} \text{atality}$ and $4,000 \text{€/injury}$. The expected owner loss due to fatalities and injuries caused by a fire occurrence is thus $\mu_o = (7.85 \cdot 10^{-3} \cdot 0.1 + 0.214 \cdot 0.004) \cdot 150 = 0.0246 \text{mill €}$. The total annual gain before tax for the RoRo-ferry is $3.075 \text{mill €/y}$ of which 30% is allocated to cover the fire risk. Inserting these values into the acceptance rule (13) using $\rho = 0.3$ gives

$$
\frac{1.2 \cdot 10^{-2} \cdot 0.0246}{0.3 \cdot 3.075} = 3.20 \cdot 10^{-3} \leq \left(1 + \frac{1}{\rho} \frac{22.2 - 0.0246}{0.0246}\right)^{-1} = 3.32 \cdot 10^{-3}
$$

(20)

Thus the public acceptance criterion (13) is satisfied. For $\rho = 1$ the criterion is the so-called LQI (or LQTAI) criterion which is less restrictive than if $\rho < 1$. For $\rho = 1$ the right side of (20) becomes $1.10 \cdot 10^{-2}$. 

Nature preservation willingness index

\[ W(x, y) = I(x, y)W \]

\[ W = GDP \times Q \]

\[ Q = \frac{T}{E} \]

\[ \frac{dp}{p} = -\frac{dQ}{Q} \]

\[ d[I(x, y)p] = -I(x, y)p \frac{dQ}{Q} \]

\[ \mu_p = \gamma^{-1}(1 - e^{-\gamma E|dQ|})p_{\min} \frac{|dQ|}{Q} \int_{\text{all } x} d(x) I[x, y(x)] dx \]

\[ I(x, y) = \exp[-|x - y|^2/(2\sigma^2)] \]